J Glob Optim (2010) 46:395–422
DOI 10.1007/s10898-009-9432-0

Approximation of optimal feedback control: a dynamic programming approach

Bao-Zhu Guo · Tao-Tao Wu

Received: 1 October 2008 / Accepted: 4 May 2009 / Published online: 18 May 2009

Abstract We consider the general continuous time finite-dimensional deterministic system under a finite horizon cost functional. Our aim is to calculate approximate solutions to the optimal feedback control. First we apply the dynamic programming principle to obtain the evolutive Hamilton–Jacobi–Bellman (HJB) equation satisfied by the value function of the optimal control problem. We then propose two schemes to solve the equation numerically. One is in terms of the time difference approximation and the other the time-space approximation. For each scheme, we prove that (a) the algorithm is convergent, that is, the solution of the discrete scheme converges to the viscosity solution of the HJB equation, and (b) the optimal control of the discrete system determined by the corresponding dynamic programming is a minimizing sequence of the optimal feedback control of the continuous counterpart. An example is presented for the time-space algorithm; the results illustrate that the scheme is effective.

Keywords Viscosity solution · Hamilton–Jacobi–Bellman equation · Finite difference · Optimal feedback control

AMS Subject Classifications 49J15 · 49L20 · 49L25 · 49Mxx

This work was supported by the National Natural Science Foundation of China and the National Research Foundation of South Africa.

B.-Z. Guo (✉) · T.-T. Wu
Academy of Mathematics and Systems Science, Academia Sinica, Beijing 100190,
People's Republic of China
e-mail: bzguo@iss.ac.cn

B.-Z. Guo
School of Computational and Applied Mathematics, University of the Witwatersrand, Wits 2050,
Johannesburg, South Africa

T.-T. Wu
Graduate University of the Chinese Academy of Sciences, Beijing 100049, People's Republic of China

1 Introduction

The celebrated Pontryagin maximum principle [31] is very effective in solving many optimal control problems. The necessary condition provided by the principle, combined with the existence and uniqueness of the optimal control in many practical engineering problems implies that we can obtain the optimal control by solving a two-point boundary-value problem.

Unfortunately, the optimal control obtained by the maximum principle is usually not in feedback form. More seriously, "these sophisticated necessary conditions rarely give an insight into the structure of the optimal control" [27]. Nevertheless, the maximum principle provides a possibility of seeking numerically the solution of the optimal control. The direct and indirect methods [30] are regarded as two basic numerical methods of solving optimal control problems through necessary conditions. For indirect method that is mainly the multiple shooting method [29], the optimal control is sought through solving a two-point boundary-value problem obtained by the Pontryagin maximum principle. However, the multiple shooting method may come up against the difficulty of "initial guess" [5]. And by direct approach, the optimal control problem is transformed into a nonlinear programming problem, and then solved by using either a penalty function method or other methods such as sequential mathematical programming methods. We refer to [26,32,35] for control parameterization method or the so-called direct shooting method. Although there is no "initial guess" problem for direct method, the simplification for the original problem leads to the fall of reliability and accuracy, and when the degree of discretization and parameterization is very high, the solving process gives rise to "curse of dimensionality" [5].

In contrast to maximum principle that deals with only one extremal problem, the Bellman dynamic programming method, on the other hand, deals with a family of extremals. Once the Hamilton–Jacobi–Bellman (HJB) equation satisfied by the value function is established, the optimal feedback control law can be found in terms of the solution to this first order nonlinear partial differential equation [1].

However, the HJB equation may have no classical solution no matter how smooth its coefficients are, a fact well-known since Pontryagin's time. To overcome this difficulty, Lions and Crandall introduced the notion of viscosity solution in the 1980s [9–12,25]. Under this weak notion, the existence and uniqueness of the solution to HJB equation is guaranteed. We refer to [4,6,18,19,24,37] for studies of the viscosity solution for infinite-dimensional optimal control problems.

This paper considers finite-dimensional control systems. When it is difficult (or impossible) to analytically solve the HJB equation, as it is usually the case, one has to seek numerical solutions instead.

Note that in seeking the viscosity solution of the HJB equation, additional information is needed during the process. Specifically, we need to have the gradient of the value function. For the purpose of numerical solution, some specially defined differences can be used to replace the required gradient that usually does not exist in the classical sense. This replacement was justified by our simulation practice [19,20] for some large inertial systems. Furthermore it was shown in [17] that some simple difference scheme may be used to produce numerically the approximation of the viscosity solution. More recent efforts in solving numerically the HJB equation for finite-dimensional control problems can be found in [7,8,13–16,22,23,28,33,34].

The optimal feedback control law is typically constructed numerically in two steps. The first step obtains an approximation to the viscosity solution of HJB equation, which can be done by either a time-discretization scheme or a time-space discretization scheme. The optimal feedback control can then be constructed based on the approximation of viscosity solution, if the convergence of the approximation is proved.

For finite-dimensional optimal control problems with infinite horizon costs, some results on the approximations of viscosity solution and optimal feedback control are available in literature, we refer to [2,7,8,15,36]. However, the *finite* horizon case is more difficult because of the way that the time is involved in the HJB equation. Although some progress have also been made in approximation of viscosity solution, see for instance, [3] and [16,17], the convergence of the approximate solution depends strongly on the introduction of a discount factor in the cost functional [16] and there are no available convergence results for the optimal controls.

In this paper, we consider a general deterministic system with finite horizon cost functional without discount factor. We first discretize directly the HJB equation by a time-discretization scheme. We then prove, for the time-discretization scheme, some properties introduced in [3], which are very important to the convergence of the approximate solution of HJB equation [17]. Using the piecewise linear interpolation, we develop an algorithm for the approximation of the viscosity solution. Based on the approximation of viscosity solution, an "almost optimal" feedback control law associated with the non-autonomous finite horizon deterministic problem is then constructed.

For a given $T > 0$, we consider the following nonlinear finite-dimensional control system:

$$\begin{cases} y'(t) = f(y(t), t, u(t)), \quad t \in (0, T], \\ y(0) = z, \end{cases} \tag{1.1}$$

where $u(\cdot) \in \Delta \triangleq L^{\infty}([0, T]; \mathbb{U})$ is the control, $\mathbb{U} \subset \mathbb{R}^m$ is a compact subset, $z \in \mathbb{R}^n$ is the initial value, $y(t) \in \mathbb{R}^n$ is the state at time t. The cost functional J is defined by

$$J(u(\cdot)) = \int_0^T L(y(t), t, u(t))\,\mathrm{d}t + \psi(y(T)), \tag{1.2}$$

where L defines the running cost, ψ is the final cost. The optimal control problem is to find $u^*(\cdot) \in \Delta$ such that

$$J\left(u^*(\cdot)\right) = \inf_{u(\cdot)\in\Delta} J(u(\cdot)). \tag{1.3}$$

The dynamic programming method is to consider the following family of optimal control problems, that is, each system starts from $(x, s) \in \mathbb{R}^n \times [0, T)$ such that

$$\begin{cases} y'(t) = f(y(t), t, u(t)), \quad t \in (s, T], \\ y(s) = x, \end{cases} \tag{1.4}$$

where $u(\cdot) \in \Delta_s \triangleq L^{\infty}([s, T]; \mathbb{U})$. The objective is to find $u_*(\cdot) \in \Delta_s$ that minimizes the cost functional of the following

$$J_{x,s}(u(\cdot)) = \int_s^T L(y(t), t, u(t))\,\mathrm{d}t + \psi(y(T)). \tag{1.5}$$

The function w defined by

$$w(x, s) = J_{x,s}(u_*(\cdot)) = \inf_{u(\cdot)\in\Delta_s} J_{x,s}(u(\cdot)) \tag{1.6}$$

is called the value function of the optimal control problem of (1.4)–(1.6). The dynamic programming principle claims that if $w \in C^1(\mathbb{R}^n \times [0, T])$ then it satisfies the following HJB equation:

$$\begin{cases} -\dfrac{\partial w}{\partial s}(x, s) - \inf\limits_{u\in\mathbb{U}}\{\nabla_x w(x, s) \cdot f(x, s, u) + L(x, s, u)\} = 0, \quad (x, s) \in \mathbb{R}^n \times [0, T), \\ w(x, T) = \psi(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.7}$$

Let us first recall the definition of viscosity solution and some properties of the value function that we need in the follows.

Definition 1.1 A function $w \in C(\mathbb{R}^n \times [0, T])$ is a sub-viscosity solution of (1.7) if for any $x \in \mathbb{R}^n$, $w(x, T) \le \psi(x)$, and for any test function $v \in C^1(\mathbb{R}^n \times [0, T))$, if $w - v$ attains strictly a local maximum at $(x^*, s^*) \in \mathbb{R}^n \times [0, T)$ then

$$-v_s\left(x^*, s^*\right) - \inf_{u\in\mathbb{U}}\left\{v_x\left(x^*, s^*\right) \cdot f\left(x^*, s^*, u\right) + L\left(x^*, s^*, u\right)\right\} \le 0.$$

Similarly, a function $w \in C(\mathbb{R}^n \times [0, T])$ is a super-viscosity solution of (1.7) if for any $x \in \mathbb{R}^n$, $w(x, T) \ge \psi(x)$, and for any test function $v \in C^1(\mathbb{R}^n \times [0, T))$, if $w - v$ attains strictly a local minimum at $(x^*, s^*) \in \mathbb{R}^n \times [0, T)$ then

$$-v_s\left(x^*, s^*\right) - \inf_{u\in\mathbb{U}}\left\{v_x\left(x^*, s^*\right) \cdot f\left(x^*, s^*, u\right) + L\left(x^*, s^*, u\right)\right\} \ge 0.$$

w is called a viscosity solution of (1.7), if w is both sub-viscosity solution and super-viscosity solution of (1.7).

The following Theorem 1.1 is a direct consequence of [11,36].

Theorem 1.1 *For system* (1.1)–(1.3)*, assume that*

$$\begin{cases} \|f(x, t, u) - f(y, s, u)\| \le L_f(\|x - y\| + |t - s|), \quad \forall\, x, y \in \mathbb{R}^n, t, s \in [0, T],\ u \in \mathbb{U}, \\ \|f(x, t, u)\| \le M_f, \quad f(x, t, \cdot) \in C(\mathbb{U}), \quad \forall\, x \in \mathbb{R}^n, t \in [0, T],\ u \in \mathbb{U}, \\ |L(x, t, u) - L(y, s, u)| \le L_L(\|x - y\| + |t - s|), \quad \forall\, x, y \in \mathbb{R}^n, t, s \in [0, T],\ u \in \mathbb{U}, \\ |L(x, t, u)| \le M_L,\ L(x, t, \cdot) \in C(\mathbb{U}), \quad \forall\, x \in \mathbb{R}^n,\ t \in [0, T],\ u \in \mathbb{U}, \\ |\psi(x) - \psi(y)| \le L_\psi\|x - y\|, \quad \forall\, x, y \in \mathbb{R}^n, \\ |\psi(x)| \le M_\psi, \quad \forall\, x \in \mathbb{R}^n, \end{cases} \tag{1.8}$$

where $L_f, M_f, L_L, M_L, L_\psi, M_\psi$ are positive constants. Then the function w defined by (1.6) *belongs to $BUC(\mathbb{R}^n \times [0, T])$, the space of bounded uniformly continuous functions on $\mathbb{R}^n \times [0, T]$, and is the unique viscosity solution of the HJB equation* (1.7)*. Moreover, there exists a constant $L_w > 0$ such that for any $(x, t), (y, s) \in \mathbb{R}^n \times [0, T]$,*

$$|w(x, t) - w(y, s)| \le L_w(\|x - y\| + |t - s|).$$

Our idea of construction of approximation scheme for viscosity solution and optimal feedback control comes from the time-discretization dynamic programming principle. We first design a time finite difference scheme for the HJB equation and show that it does satisfy the time-discretization dynamic programming principle of the corresponding discrete control system. Then a time-space discretization scheme through regular space triangulation for a closed bounded polyhedron is constructed.

The main contributions of this paper are: (a) the time and time-space approximation schemes designed are, to our best knowledge, the first such schemes that satisfy the time-discretization dynamic programming principe for finite horizon problem without discount factor, and the approximate viscosity solution is shown to be convergent to the continuous counterpart from the operator point of view. (b) the difference between the time approximate viscosity solution and time-space approximate viscosity solution constructed through a suitable space discretion is rigorously estimated, which plays an important role in the proof of the optimality for the time-space discretization control system and the convergence of the "almost optimal" feedback control constructed by the time-space approximate viscosity solution.

We proceed as follows. In next section, Sect. 2, we construct a time-discretization scheme for the HJB equation (1.7) and show the convergence of the approximation of viscosity solution to the viscosity solution of associated HJB equation. The time-space discretization and the proof of convergence are presented in Sect. 3. In Sect. 4, we synthesize "almost optimal" feedback control laws for both time-discretization and time-space discretization systems, and show that each feedback control law is a minimizing sequence of an optimal feedback control law for the original continuous control system. Some convergence rates are also given at the same time. In Sect. 5, an algorithm is streamlined and the numerical result is presented to verify the effectiveness of the algorithm. Finally, we give some concluding remarks in Sect. 6.

2 Time-discretization scheme

For each given large positive integer N, subdivide $[0, T]$ into N equal sub-intervals. Let

$$s_j = jh, \quad j = 0, 1, \ldots, N, \quad h = \frac{T}{N} < 1.$$

When $s = s_{j+1}$, $j = 0, 1, \ldots, N-1$, we discretize the time derivative term in HJB equation (1.7) by the time backward finite difference:

$$w_s(x, s_{j+1}) \cong \frac{w(x, s_{j+1}) - w(x, s_j)}{h},$$

and discretize the gradient term by the finite difference:

$$\nabla_x w(x, s_{j+1}) \cdot f(x, s_{j+1}, u) \cong \frac{w(x + hf(x, s_{j+1}, u), s_{j+1}) - w(x, s_{j+1})}{h}.$$

We then obtain the following time-discretization HJB equation:

$$\begin{cases} w_h(x, s_j) = \min\limits_{u \in \mathbb{U}} \left\{hL(x, s_{j+1}, u) + w_h\left(x + hf(x, s_{j+1}, u), s_{j+1}\right)\right\}, \\ \qquad\qquad\quad x \in \mathbb{R}^n, j = 0, 1, \ldots, N-1, \\ w_h(x, T) = \psi(x), \quad x \in \mathbb{R}^n. \end{cases}$$

Modify the above scheme for all $s \in [0, T)$, to obtain

$$\begin{cases} w_h(x, s) = \min\limits_{u \in \mathbb{U}} \left\{(s_{j+1} - s)L(x, s, u) + w_h\left(x + (s_{j+1} - s)f(x, s, u), s_{j+1}\right)\right\}, \\ \qquad\qquad\quad x \in \mathbb{R}^n, \ s \in [s_j, s_{j+1}), \ j = 0, 1, \ldots, N-1, \\ w_h(x, T) = \psi(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{2.1}$$

Before going for estimations of scheme (2.1), we need some notations. Define set I and a family of functions $\{g_h(\cdot),\ h \in I\}$, respectively, by

$$I \triangleq \{h \mid h = T/N,\ N \in \mathbb{N},\ N \geq [T] + 1\},$$

and

$$g_h(s) = \begin{cases} [s/h], & s \in [0, T), \\ N - 1, & s = T, \end{cases}$$

where $[a] \in \mathbb{N}$ denotes the largest integer that is not greater than $a \in \mathbb{R}$.

For brevity in notation, we also use $j_{h,s}$ to denote $g_h(s)$ for all $s \in [0, T]$. By the definition of g_h, we know that $j_{h,s}$ is the unique integer such that $s \in [s_{j_{h,s}}, s_{j_{h,s}+1})$. Obviously, $I \subset (0, 1)$ and $j_{h,s} \in \{0, 1, \ldots, N-1\}$ for all $s \in [0, T]$. Note that $N = \frac{T}{h}$ increases as long as $h \in I$ decreases.

Proposition 2.1 *Suppose that all hypotheses of Theorem 1.1 hold. Then there exists a unique solution w_h to (2.1), and for any $x, y \in \mathbb{R}^n$, $j = 0, 1, \ldots, N-1$, the following estimates hold true:*

- (i) $|w_h(x, s_j)| \leq (N - j)hM_L + M_\psi$.
- (ii) $|w_h(x, s_j) - w_h(y, s_j)| \leq C_j \|x - y\|$, $C_j \triangleq (TL_L + L_\psi)(1 + hL_f)^{N-j} \leq C_{\max} \triangleq (TL_L + L_\psi)e^{L_f}$.
- (iii) $|w_h(x, s_{j+1}) - w_h(x, t)| \leq \left(M_L + C_{j+1}M_f\right)|s_{j+1} - t|$ *for any* $t \in [s_j, s_{j+1})$.
- (iv) $|w_h(x, s) - w_h(x, t)| \leq [M_L + L_L + C_{j+1}(M_f + L_f)]|s - t|$ *for any* $s, t \in [s_j, s_{j+1})$.
- (v) $\{w_h,\ h \in I\}$ *is uniformly bounded and equi-continuous in* $\mathbb{R}^n \times [0, T]$. *Precisely,* $|w_h(x, s)| \leq TM_L + M_\psi$ *for any* $s \in [0, T]$, $|w_h(x, s) - w_h(y, s)| \leq [L_L + (1 + L_f)C_{\max}]\|x - y\|$ *for any* $s \in [0, T]$, $|w_h(x, s) - w_h(x, t)| \leq [M_L + L_L + C_{\max}(M_f + L_f)]|s - t|$ *for any* $s, t \in [0, T]$.

Proof The proof of the existence and uniqueness of w_h is trivial and omitted. We now prove the properties (i)–(v).

(i) This can be done step by step. Actually, it follows directly from (2.1) that for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
&|w_h(x, s_N)| = |\psi(x)| \leq M_\psi, \\
&|w_h(x, s_{N-1})| \leq hM_L + |w_h(x + hf(x, s_{N-1}, u), s_N)| \leq hM_L + M_\psi, \\
&|w_h(x, s_{N-2})| \leq hM_L + |w_h(x + hf(x, s_{N-2}, u), s_{N-1})| \leq 2hM_L + M_\psi, \\
&\ldots \\
&|w_h(x, s_j)| \leq hM_L + |w_h(x + hf(x, s_j, u), s_{j+1})| \leq (N - j)hM_L \\
&\qquad\qquad + M_\psi,\ \ j = N - 3, N - 4, \ldots, 1, 0,
\end{aligned}$$

proving (i).

(ii) This can be proved similarly as (i). Actually, for any $x, y \in \mathbb{R}^n$,

$$|w_h(x, s_N) - w_h(y, s_N)| = |\psi(x) - \psi(y)| \leq L_\psi \|x - y\| \triangleq \overline{C}_N \|x - y\|,$$

$$\begin{aligned}
&|w_h(x, s_{N-1}) - w_h(y, s_{N-1})| \\
&\leq \max_{u \in \mathbb{U}} \Big| \big[w_h\left(x + hf(x, s_{N-1}, u), s_N\right) - w_h\left(y + hf(y, s_{N-1}, u), s_N\right)\big] \\
&\quad + h\left[L(x, s_{N-1}, u) - L(y, s_{N-1}, u)\right]\Big| \\
&\leq \max_{u \in \mathbb{U}} \left\{\overline{C}_N \left\|x - y + h\left[f(x, s_{N-1}, u) - f(y, s_{N-1}, u)\right]\right\| + hL_L\|x - y\|\right\} \\
&\leq \left[\overline{C}_N(1 + hL_f) + hL_L\right]\|x - y\| \triangleq \overline{C}_{N-1}\|x - y\|,
\end{aligned}$$

and

$$
\begin{aligned}
&\left|w_h(x, s_j) - w_h(y, s_j)\right| \\
&\quad \le \max_{u\in\mathbb{U}} \left|\left[w_h\left(x + hf(x, s_j, u), s_{j+1}\right) - w_h\left(y + hf(y, s_j, u), s_{j+1}\right)\right]\right. \\
&\qquad \left. + h\left[L(x, s_j, u) - L(y, s_j, u)\right]\right| \\
&\quad \le \max_{u\in\mathbb{U}} \left\{\overline{C}_{j+1}\left\|x - y + h\left[f(x, s_j, u) - f(y, s_j, u)\right]\right\| + hL_L\|x-y\|\right\} \\
&\quad \le \left[\overline{C}_{j+1}(1 + hL_f) + hL_L\right]\|x-y\| \triangleq \overline{C}_j\|x-y\|, \quad j = N-2, N-3, \ldots, 1, 0,
\end{aligned}
$$

that is,

$$
\overline{C}_N = L_\psi, \quad \overline{C}_j = hL_L + (1 + hL_f)\overline{C}_{j+1}, \quad j = N-1, N-2, \ldots, 1, 0.
$$

Solving the above backward difference equation gives (ii):

$$
\begin{aligned}
\overline{C}_j &= \sum_{l=0}^{N-j-1} (1 + hL_f)^l hL_L + (1 + hL_f)^{N-j} L_\psi \le (N-j)hL_L(1 + hL_f)^{N-j} \\
&\quad + (1 + hL_f)^{N-j} L_\psi \\
&\le \left[TL_L + L_\psi\right](1 + hL_f)^{N-j} \triangleq C_j \le C_{\max} \triangleq (TL_L + L_\psi)e^{L_f}.
\end{aligned}
$$

(iii) For any $j \in \{0, 1, \ldots, N-1\}$, if $t \in [s_j, s_{j+1})$, it follows from (ii) that for any $x \in \mathbb{R}^n$,

$$
\begin{aligned}
\left|w_h(x, s_{j+1}) - w_h(x, t)\right| &\le \max_{u\in\mathbb{U}} \left| w_h(x, s_{j+1}) - (s_{j+1} - t)L(x, t, u)\right. \\
&\qquad \left. - w_h\left(x + (s_{j+1} - t)f(x, t, u), s_{j+1}\right)\right| \\
&\le [M_L + C_{j+1}M_f]|s_{j+1} - t|.
\end{aligned}
$$

This is (iii).

(iv) Now we prove (iv). Similar to (iii), for any $j \in \{0, 1, \ldots, N-1\}$, if $s, t \in [s_j, s_{j+1})$, we have, for any $x \in \mathbb{R}^n$, that

$$
\begin{aligned}
|w_h(x, s) - w_h(x, t)| &\le \max_{u\in\mathbb{U}} |\,(t-s)L(x, s, u) + (s_{j+1} - t)\left[L(x, s, u) - L(x, t, u)\right] \\
&\qquad + w_h\left(x + (s_{j+1} - s)f(x, s, u), s_{j+1}\right) \\
&\qquad - w_h\left(x + (s_{j+1} - t)f(x, t, u), s_{j+1}\right)| \\
&\le (M_L + hL_L)|s-t| + C_{j+1}(M_f + hL_f)|s-t| \\
&\le \left[M_L + L_L + C_{j+1}(M_f + L_f)\right]|s-t|.
\end{aligned}
$$

(iv) is proved.

(v) First, we claim that $\{w_h,\ h \in I\}$ is uniformly bounded in $\mathbb{R}^n \times [0, T]$. Actually, when $s = T$, $|w_h(x, T)| = |\psi(x)| \le M_\psi, \quad \forall\, x \in \mathbb{R}^n$, and for any $(x, s) \in \mathbb{R}^n \times [0, T)$,

$$
\begin{aligned}
|w_h(x, s)| &\le (s_{j_{h,s}+1} - s)M_L + |w_h(x + (s_{j_{h,s}+1} - s)f(x, s, u), s_{j_{h,s}+1})| \\
&\le [1 + (N - j_{h,s} - 1)]hM_L + M_\psi \le TM_L + M_\psi.
\end{aligned}
$$

Next, when $s = T$, it has

$$
|w_h(x, T) - w_h(y, T)| = |\psi(x) - \psi(y)| \le L_\psi\|x-y\|, \quad \forall\, x, y \in \mathbb{R}^n,
$$

and for any $s \in [0, T)$,

$$\begin{aligned}
&|w_h(x,s) - w_h(y,s)| \\
&\quad \le \max_{u\in\mathbb{U}} \Big\{ \big| w_h\left(x + (s_{j_{h,s}+1} - s) f(x,s,u), s_{j_{h,s}+1}\right) \\
&\qquad - w_h\left(y + (s_{j_{h,s}+1} - s) f(y,s,u), s_{j_{h,s}+1}\right) \big| \\
&\qquad + (s_{j_{h,s}+1} - s) \left| L(x,s,u) - L(y,s,u) \right| \Big\} \\
&\quad \le h L_L \|x-y\| + C_{j_{h,s}+1} \left(\|x-y\| + h L_f \|x-y\| \right) \\
&\quad \le \left[L_L + (1+L_f) C_{\max} \right] \|x-y\|, \quad \forall\, x, y \in \mathbb{R}^n.
\end{aligned}$$

Thus for any $s \in [0, T]$, $\{w_h(\cdot, s),\ h \in I\}$ is equi-continuous in $\mathbb{R}^n$. What remains to be shown is that for any $x \in \mathbb{R}^n$, $\{w_h(x, \cdot),\ h \in I\}$ is also equi-continuous in $[0, T]$. To do this, we proceed it as follows.

By (iii)–(iv), we only need to consider the case that $s, t \in [0, T]$ are in different intervals, that is, $s \in [s_{j_{h,s}}, s_{j_{h,s}+1})$, $t \in [s_{j_{h,t}}, s_{j_{h,t}+1})$ with $0 \le j_{h,s} < j_{h,t} \le N-1$. For any $x \in \mathbb{R}^n$, from (iii)–(iv), we have

$$\begin{aligned}
&|w_h(x,s) - w_h(x,t)| \\
&= \left| w_h(x,s) - w_h(x, s_{j_{h,s}+1}) + \sum_{l=j_{h,s}+1}^{j_{h,t}-1} [w_h(x,s_l) - w_h(x,s_{l+1})] + w_h(x, s_{j_{h,t}}) - w_h(x,t) \right| \\
&\le \left| w_h(x,s) - w_h(x, s_{j_{h,s}+1}) \right| + \sum_{l=j_{h,s}+1}^{j_{h,t}-1} \left| w_h(x,s_l) - w_h(x,s_{l+1}) \right| + \left| w_h(x, s_{j_{h,t}}) - w_h(x,t) \right| \\
&\le C^* \left(|s - s_{j_{h,s}+1}| + \sum_{l=j_{h,s}+1}^{j_{h,t}-1} |s_l - s_{l+1}| + |s_{j_{h,t}} - t| \right) \quad \le C^* |s-t|,
\end{aligned}$$

where $C^* = \max_{l=0,1,\ldots,N-1}\{M_L + C_l M_f,\ M_L + L_L + C_{l+1}(M_f + L_f)\} = M_L + L_L + C_{\max}(M_f + L_f)$.

Therefore, $\{w_h,\ h \in I\}$ is uniformly bounded and equi-continuous in $\mathbb{R}^n \times [0, T]$. The proof is complete. □

Before proving the convergence of w_h, we need some additional concepts and a preliminary result.

For any $s \in [0, T)$, let $p_h(s) = s_{j_{h,s}+1} - s$. Define operator $F_h(s, p_h(s)) : C(\mathbb{R}^n) \to C(\mathbb{R}^n)$ by

$$F_h(s, p_h(s))[\phi(\cdot)](x) = \min_{u\in\mathbb{U}} \{p_h(s) L(x,s,u) + \phi(x + p_h(s) f(x,s,u))\}, \quad \forall\, \phi \in C(\mathbb{R}^n). \tag{2.2}$$

For brevity in notation, we also use F_h to denote $F_h(s, p_h(s))$. In this way, the time-discretization scheme (2.1) can be rewritten as

$$\begin{cases} w_h(x,s) = F_h\left[w_h(\cdot, s_{j+1})\right](x), & x \in \mathbb{R}^n,\ s \in [s_j, s_{j+1}),\ j = 0, 1, \ldots, N-1, \\ w_h(x,T) = \psi(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.3}$$

The properties of F_h that will be used later are stated in Lemma 2.1 below.

Lemma 2.1 *Let F_h be defined by (2.2). Then*

(i) *Monotonicity:*
For any $(x, s) \in \mathbb{R}^n \times [0, T)$, $\phi_1, \phi_2 \in C(\mathbb{R}^n)$, *if for any* $u \in \mathbb{U}$,

$$\phi_1(x + p_h(s) f(x, s, u)) \leq \phi_2(x + p_h(s) f(x, s, u)),$$

then

$$F_h[\phi_1](x) \leq F_h[\phi_2](x).$$

(ii) *Constant translation property:*

$$F_h[\phi + c] = F_h[\phi] + c, \quad \forall\, \phi \in C(\mathbb{R}^n),\ c \in \mathbb{R}.$$

(iii) *Consistency:*
For any $(x, t) \in \mathbb{R}^n \times [0, T)$ *and any test function* $v \in C^1(\mathbb{R}^n \times [0, T))$,

$$\lim_{\substack{(y,s)\to(x,t)\\ h\to 0^+}} p_h(s)^{-1}\left\{F_h\left[v(\cdot, s_{j_{h,s}+1})\right](y) - v(y, s)\right\}$$
$$= v_t(x, t) + \inf_{u\in\mathbb{U}} \{v_x(x, t) \cdot f(x, t, u) + L(x, t, u)\},$$

where $p_h(s) = s_{j_{h,s}+1} - s$.

Proof Since (i) and (ii) are the direct consequences of definition, we only need to prove (iii). Now

$$F_h(s, p_h(s))\left[v(\cdot, s_{j_{h,s}+1})\right](y) - v(y, s)$$
$$= \min_{u\in\mathbb{U}}\left\{p_h(s)L(y, s, u) + v\left(y + p_h(s) f(y, s, u), s_{j_{h,s}+1}\right) - v(y, s_{j_{h,s}+1})\right.$$
$$\left. + v(y, s_{j_{h,s}+1}) - v(y, s)\right\}$$
$$= p_h(s) \cdot \left[v_s(y, \bar{s}) + \min_{u\in\mathbb{U}}\left\{v_y(\bar{y}, s_{j_{h,s}+1}) \cdot f(y, s, u) + L(y, s, u)\right\}\right],$$

where $\bar{s} = s + \theta_1 \cdot p_h(s)$, $\bar{y} = y + \theta_2 \cdot p_h(s) f(y, s, u)$ for some $\theta_1,\ \theta_2 \in (0, 1)$. We see that when $h \to 0^+$, it has $(y, s) \to (x, t)$, and hence $p_h(s) \to 0^+$, $(\bar{y}, \bar{s}) \to (x, t)$, $s_{j_{h,s}+1} \to t$. The conclusion then follows from the fact $v \in C^1(\mathbb{R}^n \times [0, T))$. □

Theorem 2.1 *Suppose that all hypotheses of Theorem 1.1 hold. Let* w_h *be the solution of* (2.1) *(or* (2.3)*) and* w *be the viscosity solution of* (1.7)*. Then for any compact subset* $Q \subset \mathbb{R}^n$,

$$\lim_{h\to 0^+} w_h = w \quad \text{uniformly on } Q \times [0, T].$$

Proof Since from (v) of Proposition 2.1, $\{w_h,\ h \in I\}$ is a family of uniformly bounded and equi-continuous functions on $\mathbb{R}^n \times [0, T]$, by the Arezela–Ascoli theorem, for any compact subset $Q \subset \mathbb{R}^n$, there is a subsequence of $\{w_h,\ h \in I\}$ (that we still denote by $\{w_h\}$ without confusion) such that w_h converges to a function w^* uniformly on $Q \times [0, T]$ as $h \to 0^+$. We show that w^* is a viscosity solution of (1.7).

For any test function $v \in C^1\,(\mathbb{R}^n \times [0, T))$, suppose that $(x^*, s^*) \in \mathbb{R}^n \times [0, T)$ is a strict local maximum point of $w^* - v$. Then there exists a closed ball $B = B\,(x^*, s^*)$ of $\mathbb{R}^n \times [0, T)$ centered at (x^*, s^*) such that

$$w^*\left(x^*, s^*\right) - v\left(x^*, s^*\right) > w^*(x, s) - v(x, s), \quad \forall\, (x, s) \in B\backslash\left(x^*, s^*\right).$$

Then there exists a sequence $\{(x_h, s_h)\}$ in B such that $(x_h, s_h) \in B$ is a maximum point for $w_h - v$ on B, and

$$x_h \to x^* \quad \text{and} \quad s_h \to s^* \quad \text{as} \quad h \to 0^+.$$

Note that $s_h \in [s_{j_{h,s_h}}, s_{j_{h,s_h}+1})$. Define $p_h = s_{j_{h,s_h}+1} - s_h$. Obviously, $p_h > 0$ and as long as $h > 0$ small enough, $j_{h,s_h} < N - 1$ and $(x_h + p_h f(x_h, s_h, u), s_h + p_h) \in B$ for all $u \in \mathbb{U}$.

With above preparations, for any $u \in \mathbb{U}$, we have

$$w_h(x_h, s_h) - v(x_h, s_h) \geq w_h(x_h + p_h f(x_h, s_h, u), s_h + p_h) - v(x_h + p_h f(x_h, s_h, u),\\ s_h + p_h),$$

that is,

$$v(x_h + p_h f(x_h, s_h, u), s_h + p_h) - v(x_h, s_h) \geq w_h(x_h + p_h f(x_h, s_h, u), s_h + p_h)\\ - w_h(x_h, s_h)$$

or

$$\phi_1(x_h + p_h f(x_h, s_h, u)) \geq \phi_2(x_h + p_h f(x_h, s_h, u)), \quad \forall\, u \in \mathbb{U},$$

where $\phi_1, \phi_2 \in C^1(\mathbb{R}^n)$ are defined as

$$\phi_1(x) \triangleq v(x, s_h + p_h) - v(x_h, s_h), \;\; \phi_2(x) \triangleq w_h(x, s_h + p_h) - w_h(x_h, s_h), \quad x \in \mathbb{R}^n.$$

Apply the operator $F_h(s_h, p_h)$ to ϕ_1, ϕ_2 and take the monotonicity and constant translation property of F_h justified by Lemma 2.1 into account, to get

$$F_h\left[v(\cdot, s_h + p_h)\right](x_h) - v(x_h, s_h) \geq F_h\left[w_h(\cdot, s_h + p_h)\right](x_h) - w_h(x_h, s_h) = 0.$$

Divided by p_h on both sides above, pass the limit of $h \to 0^+$ and use (iii) of Lemma 2.1, to obtain

$$v_s\left(x^*, s^*\right) + \inf_{u \in \mathbb{U}} \left\{ v_x\left(x^*, s^*\right) \cdot f\left(x^*, s^*, u\right) + L\left(x^*, s^*, u\right) \right\} \geq 0,$$

which means that w^* is a sub-viscosity solution of (1.7) by noting the fact $w_h(\cdot, T) = \psi(\cdot)$ on $\mathbb{R}^n$.

Along the same line, we can also show that w^* is a super-viscosity solution of (1.7). Therefore, w^* is a viscosity solution of (1.7). The result then follows from the uniqueness of viscosity solution of (1.7). □

3 Space-discretization scheme

In this section, we consider a space approximation of w_h. For simplicity, we make a standard assumption used in [15] that $\Omega \subset \mathbb{R}^n$ is a closed bounded polyhedron such that for all sufficiently small h,

$$x + hf(x, s, u) \in \Omega, \quad \forall\, (x, s, u) \in \Omega \times [0, T] \times \mathbb{U}. \tag{3.1}$$

We construct a regular triangulation of Ω consisting of a finite number of simplices $\{T_i\}$ such that $\Omega = \cup_i T_i$. Denote by x_j, $j = 1, 2, \ldots, M$, the nodes of the triangulation. Set

$$k = \max_i \{\text{diam}(T_i)\}, \quad S = \{s_j \mid j = 0, 1, \ldots, N\}, \quad G = \{x_j \mid j = 1, 2, \ldots, M\},$$

where $\text{diam}(T_i)$ denotes the diameter of the set T_i. It is clear that

$$\lim_{k \to 0^+} \text{dist}(x, G) = 0, \quad \forall\, x \in \Omega.$$

For $(x_i, s_j) \in G \times S$, define

$$\begin{cases} v_h^k(x_i, s_j) = \min_{u\in\mathbb{U}} \left\{ hL(x_i, s_j, u) + \sum_{l=1}^{M} \lambda_i^l(u, s_j) v_h^k(x_l, s_{j+1}) \right\}, \\ \qquad\qquad i = 1, 2, \dots, M,\ \ j = 0, 1, \dots, N-1, \\ v_h^k(x_i, T) = \psi(x_i), \quad i = 1, 2, \dots, M, \end{cases} \tag{3.2}$$

where $\left\{\lambda_i^l(u, s_j) \in [0, 1],\ l = 1, 2, \dots, M\right\}$ is the set of coefficients in the unique convex linear combination $x_i + hf(x_i, s_j, u) = \sum_{l=1}^{M} \lambda_i^l(u, s_j) x_l \in \Omega$ (by assumption 3.1) with $\sum_{l=1}^{M} \lambda_i^l(u, s_j) = 1$. This convex linear combination is produced in terms of the vertices of the simplex where $x_i + hf(x_i, s_j, u)$ is located.

In this way, v_h^k is well-defined on $G \times S$. Moreover, similar to Proposition 2.1, there exists a unique solution $v_h^k \in B(G \times S)$ to (3.2). Here, $B(S_0)$ denotes the space of all bounded functions on the region S_0.

Based on scheme (3.2), we define a piecewise linear function $w_h^k \in B(\Omega \times [0, T])$ by

$$\begin{cases} w_h^k(x, s) = \sum_{i=1}^{M} \lambda_i(x) \left[\mu(s) v_h^k(x_i, s_j) + (1 - \mu(s)) v_h^k(x_i, s_{j+1}) \right], \\ \qquad\qquad x \in \Omega,\ \ s \in [s_j, s_{j+1}),\ \ j = 0, \dots, N-1, \\ w_h^k(x, T) = \sum_{i=1}^{M} \lambda_i(x) \psi(x_i), \quad x \in \Omega, \end{cases} \tag{3.3}$$

where $\mu(s) \triangleq \dfrac{s_{j+1} - s}{h} \in (0, 1]$ when $s \in [s_j, s_{j+1})$, and $\{\lambda_i(x) \in [0, 1],\ i = 1, 2, \dots, M\}$ is the set of coefficients in the unique convex linear combination $x = \sum_{i=1}^{M} \lambda_i(x) x_i$ with $\sum_{i=1}^{M} \lambda_i(x) = 1$. This convex linear combination is produced also in terms of the vertices of the simplex where x is located. Obviously, $w_h^k = v_h^k$ on $G \times S$.

Theorem 3.1 *Suppose that all hypotheses of Theorem 1.1 hold. Let w_h, w_h^k be the solutions to* (2.1) and (3.3) *on $\Omega \times S$, respectively. Then we have, for any $(x, s_j) \in \Omega \times S$, that*

$$\left| w_h(x, s_j) - w_h^k(x, s_j) \right| \le C \left(\frac{k}{h} + k \right), \tag{3.4}$$

where $C \triangleq \max\{TC_{\max}, L_\psi\}$ is independent of h, k.

Proof For any $(x, s_j) \in \Omega \times S$, suppose that x lies in some simplex T_{i_x}. Let $x = \sum_{i\in I_x} \lambda_i(x) x_i$ be the unique convex linear combination which is produced in terms of the vertices of T_{i_x}, where

$$I_x \triangleq \{i \mid x_i \in T_{i_x}, \quad i = 1, 2, \dots, M\}.$$

Then

$$\|x - x_i\| \le k \quad \text{for any} \quad i \in I_x.$$

Next, by Definition 3.3, for any $u \in \mathbb{U}$ and $j \in \{0, 1, \dots, N-1\}$, it has

$$w_h^k \left(\sum_{l=1}^{M} \lambda_i^l(u, s_j) x_l, s_{j+1} \right) = \sum_{l=1}^{M} \lambda_i^l(u, s_j) w_h^k(x_l, s_{j+1}),$$

where $\left\{\lambda_i^l(u, s_j) \in [0,1],\ 1 \le l \le M\right\}$ is the set of coefficients in the unique convex linear combination $x_i + hf(x_i, s_j, u) = \sum_{l=1}^M \lambda_i^l(u, s_j)x_l$ with $\sum_{l=1}^M \lambda_i^l(u, s_j) = 1$. Once again, this convex linear combination is produced in terms of the vertices of the simplex where $x_i + hf(x_i, s_j, u)$ is located.

It follows from (ii) of Proposition 2.1 that

$$\left|w_h(x, s_j) - w_h(x_i, s_j)\right| \le C_{\max}\|x - x_i\|, \quad \forall\, i \in I_x,\ j = 0, 1, \ldots, N-1.$$

When $j = N$, it has

$$\begin{aligned}\left|w_h(x, s_N) - w_h^k(x, s_N)\right| &= \left|\psi(x) - \sum_{i\in I_x}\lambda_i(u, s_N)\psi(x_i)\right| \\ &\le \sum_{i\in I_x}\lambda_i(u, s_N)|\psi(x) - \psi(x_i)| \le L_\psi k. \end{aligned} \tag{3.5}$$

For any $j \in \{0, 1, \ldots, N-1\}$, by above justified facts and Eqs. (2.1), (3.2)–(3.3), we have, with iteration, that

$$\begin{aligned}\left|w_h(x, s_j) - w_h^k(x, s_j)\right| &\le \sum_{i\in I_x}\lambda_i(x)\left\{\left|w_h(x, s_j) - w_h(x_i, s_j)\right| + \left|w_h(x_i, s_j) - w_h^k(x_i, s_j)\right|\right\} \\ &\le C_{\max}k + \sum_{i\in I_x}\lambda_i(x)\left|w_h(x_i, s_j) - w_h^k(x_i, s_j)\right| \\ &= C_{\max}k + \sum_{i\in I_x}\lambda_i(x)\left|\min_{u\in\mathbb{U}}\left\{hL(x_i, s_j, u) + w_h(y_u, s_{j+1})\right\}\right. \\ &\quad \left. - \min_{u\in\mathbb{U}}\left\{hL(x_i, s_j, u)\sum_{l=1}^M \lambda_i^l(u, s_j)w_h^k(x_l, s_{j+1})\right\}\right| \\ &\le C_{\max}k + \sum_{i\in I_x}\lambda_i(x)\max_{u\in\mathbb{U}}\left|w_h(y_u, s_{j+1})\right. \\ &\quad \left. - \sum_{l=1}^M \lambda_i^l(u, s_j)w_h^k(x_l, s_{j+1})\right| \\ &= C_{\max}k + \sum_{i\in I_x}\lambda_i(x)\max_{u\in\mathbb{U}}\left|w_h(y_u, s_{j+1})\right. \\ &\quad \left. - w_h^k\left(\sum_{l=1}^M \lambda_i^l(u, s_j)x_l, s_{j+1}\right)\right| \\ &= C_{\max}k + \sum_{i\in I_x}\lambda_i(x)\max_{u\in\mathbb{U}}\left|w_h(y_u, s_{j+1}) - w_h^k(y_u, s_{j+1})\right| \\ &\le C_{\max}k + \sup_{z_1\in\Omega}\left|w_h(z_1, s_{j+1}) - w_h^k(z_1, s_{j+1})\right| \\ &\le 2C_{\max}k + \sup_{z_1\in\Omega}\sup_{z_2\in\Omega}\left|w_h(z_2, s_{j+2}) - w_h^k(z_2, s_{j+2})\right| \end{aligned}$$

$$
\begin{aligned}
&= 2C_{\max}k + \sup_{z_2\in\Omega}\left|w_h(z_2, s_{j+2}) - w_h^k(z_2, s_{j+2})\right| \\
&\le \ldots \\
&\le (N-j)C_{\max}k + \sup_{z_{N-j}\in\Omega}\left|w_h(z_{N-j}, s_N) - w_h^k(z_{N-j}, s_N)\right| \\
&\le (N-j)C_{\max}k + L_\psi k \le TC_{\max}\frac{k}{h} + L_\psi k \\
&\le C\left(\frac{k}{h}+k\right), \quad \text{where } C \triangleq \max\{TC_{\max}, L_\psi\}.
\end{aligned}
\tag{3.6}
$$

Notice that in the above, the convex linear combination $y_u \triangleq x_i + hf(x_i, s_j, u) = \sum_{l=1}^{M}\lambda_i^l (u, s_j)x_l$ is produced in the same way as in (3.2). The proof is completed by (3.5) and (3.6). □

Theorem 3.2 *Suppose that all hypotheses of Theorem 1.1 hold. Let $k = O(h^{1+\gamma})$ for some constant $\gamma > 0$, and let w be the viscosity solution of (1.7), w_h^k the solution of (3.3). Then*

$$\lim_{h\to 0^+} w_h^k = w \text{ uniformly on } \Omega\times[0,T].$$

Proof First, when $s = T$, by (3.4), we have, for any $x \in \Omega$, that

$$
\begin{aligned}
\left|w_h^k(x,T) - w(x,T)\right| &\le \left|w_h^k(x,T) - w_h(x,T)\right| + |w_h(x,T) - w(x,T)| \\
&= \left|w_h^k(x,T) - w_h(x,T)\right| \le C\left(\frac{k}{h}+k\right) \\
&= O\left(h^\gamma + h^{1+\gamma}\right) \to 0 \quad \text{as } h\to 0^+.
\end{aligned}
\tag{3.7}
$$

Next, for any $s \in [0,T)$, define $\mu(s) = \dfrac{s_{j_{h,s}+1} - s}{h}$. Then

$$\mu(s) \in (0,1] \quad \text{and} \quad s = \mu(s)s_{j_{h,s}} + (1-\mu(s))s_{j_{h,s}+1}.$$

By Eq. (3.3), for any $x \in \Omega$,

$$
\begin{aligned}
\left|w_h^k(x,s) - w(x,s)\right| &\le \mu(s)\left|w_h^k(x, s_{j_{h,s}}) - w(x,s)\right| \\
&\quad + (1-\mu(s))\left|w_h^k(x, s_{j_{h,s}+1}) - w(x,s)\right|,
\end{aligned}
\tag{3.8}
$$

and (we treat only $s_{j_{h,s}}$ term)

$$
\begin{aligned}
\left|w_h^k(x, s_{j_{h,s}}) - w(x,s)\right| &\le \left|w_h^k(x, s_{j_{h,s}}) - w_h(x, s_{j_{h,s}})\right| \\
&\quad + \left|w_h(x, s_{j_{h,s}}) - w_h(x,s)\right| + |w_h(x,s) - w(x,s)| \\
&\triangleq I_1 + I_2 + I_3.
\end{aligned}
\tag{3.9}
$$

By (3.4), the term

$$I_1 \triangleq \left|w_h^k(x, s_{j_{h,s}}) - w_h(x, s_{j_{h,s}})\right| \le C\left(\frac{k}{h}+k\right) = O\left(h^\gamma + h^{1+\gamma}\right) \to 0 \quad \text{as } h\to 0^+.$$

By (iv) of Proposition 2.1 and Theorem 2.1, respectively, the terms I_2 and I_3 converge to 0 uniformly on $\Omega\times[0,T]$ as $h\to 0^+$. The proof is then completed by (3.7)–(3.9). □

4 Approximation of optimal feedback control

In this section, we first develop the corresponding dynamic programming principle for the time-discretization scheme. Based on the approximations of the viscosity solution discussed in Sects. 2 and 3, we are able to construct the optimal feedback controls for the corresponding discrete control systems. The convergence of the discrete optimal feedback control to the continuous counterpart is then followed by referring to the infinite horizon case discussed in [15].

4.1 Time-discretization case

We first consider the time-discretization case. Recall that for each given large positive integer N, $[0, T]$ is subdivided into N equal sub-intervals and $s_j = jh, j = 0, 1, \dots, N$, where $h = T/N < 1$. Recall that for any $s \in [0, T)$, $j_{h,s}$ is the unique integer such that $s \in [s_{j_{h,s}}, s_{j_{h,s}+1})$. Let

$$
\begin{aligned}
&t_0 = s,\ t_l = s_{j_{h,s}+l},\ l = 1, 2, \dots, N - j_{h,s},\ h_0 = s_{j_{h,s}+1} - s,\ h_l = h,\\
&l = 1, 2, \dots, N - j_{h,s} - 1.
\end{aligned}
$$

For any $x \in \mathbb{R}^n$, consider the time-discretization counterpart for control system (1.4) which starts from (x, s):

$$
\begin{cases} y_{l+1} = y_l + h_l f(y_l, t_l, u_l), \quad l = 0, 1, \dots, N - j_{h,s} - 1,\\ y_0 = x, \end{cases} \tag{4.1}
$$

where $u_l = u_h(t_l)$, $y_l = y_{t_l}$, $u_h(\cdot) \in \Delta_s$. The cost functional $J^h_{x,s}$ and value function v_h to the discrete system (4.1) are defined, respectively, by

$$
J^h_{x,s}(u_h(\cdot)) = \sum_{l=0}^{N-j_{h,s}-1} h_l L(y_l, t_l, u_l) + \psi(y_{N-j_{h,s}}), \tag{4.2}
$$

and

$$
v_h(x, s) = \inf_{u_h(\cdot)\in\Delta_s} J^h_{x,s}(u_h(\cdot)). \tag{4.3}
$$

Lemma 4.1 [Time-discretization dynamic programming principle] *Suppose that all hypotheses of Theorem 1.1 hold. Let v_h be defined as (4.3). Then for $i = 1, 2, \dots, N - j_{h,s} - 1$,*

$$
v_h(x, s) = \min_{u_0,\dots,u_{i-1}\in\mathbb{U}} \left\{ \sum_{l=0}^{i-1} h_k L(y_l, t_l, u_l) + v_h(y_i, t_i) \right\}, \quad \forall\ (x, s) \in \mathbb{R}^n \times [0, T). \tag{4.4}
$$

Proof The proof is similar to that of Lemma 2.1 of [16]. We omit it here. □

Corollary 4.1 *Let $i = 1$ in Lemma 4.1. Then (4.4) reduces to the time-discretization scheme (2.1) in Sect. 2:*

$$
\begin{cases} v_h(x, s) = \min\limits_{u\in\mathbb{U}}\{(s_{j+1} - s)L(x, s, u) + v_h(x + (s_{j+1} - s) f(x, s, u), s_{j+1})\},\\ \qquad\qquad x \in \mathbb{R}^n,\ s \in [s_j, s_{j+1}),\ j = 0, 1, \dots, N - 1,\\ v_h(x, T) = \psi(x), \quad x \in \mathbb{R}^n. \end{cases}
$$

By Corollary 4.1, $v_h \equiv w_h$ on $\mathbb{R}^n \times [0, T]$. Due to this fact, we will use w_h instead of v_h in what follows. Consider the following subset of $\mathbb{U}$

$$\begin{aligned} A_1(x,s) = \min_{\|\cdot\|} \big\{ u \in \mathbb{U} |\ & w_h(x,s) = (s_{j+1} - s) L(x,s,u) \\ & + w_h \left(x + (s_{j+1} - s) f(x,s,u), s_{j+1} \right) \big\}, \\ & x \in \mathbb{R}^n,\ s \in [s_j, s_{j+1}), \quad j = 0, 1, \ldots, N-1. \end{aligned} \tag{4.5}$$

$A_1(x, s)$ is a subset of controls with minimal energy which satisfy the time-discretization scheme (2.1). By the assumption of (1.8), for any $(x, s) \in \mathbb{R}^n \times [0, T)$, $A_1(x, s)$ is not empty. It is remarked that $A_1(x, s)$ may not be a singleton, but we can choose any element $a_1(x, s)$ of $A_1(x, s) \subset \mathbb{U} \subset \mathbb{R}^m$, to determine the minimal energy control function m_h^* by

$$m_h^*(x,s) = a_1(x,s),\ x \in \mathbb{R}^n,\ s \in [s_j, s_{j+1}), \quad j = 0, 1, \ldots, N-1. \tag{4.6}$$

In this sense the control m_h^* is well defined on $\mathbb{R}^n \times [0, T)$.

The time-discretization counterpart for control system (1.1) is the special case of control system (4.1) with $x = z$ and $s = j_{h,s} = 0$:

$$\begin{cases} y^{j+1} = y^j + hf\left(y^j, s_j, u_h^*(s_j)\right), & j = 0, 1, \ldots, N-1, \\ y^0 = z, \end{cases} \tag{4.7}$$

in which $z \in \mathbb{R}^n$ is given and the optimal feedback control law u_h^* is taken as

$$u_h^*(t) = u_j^* \triangleq m_h^*\left(y^j, s_j\right), \quad t \in [s_j, s_{j+1}), \quad j = 0, 1, \ldots, N-1. \tag{4.8}$$

Here in order to distinguish the solution of (4.1) that starts from (x, s), we use $y^j \approx y(s_j)$ to denote the solution of system (4.7).

Now we state the convergence result.

Theorem 4.1 [Time-discretization optimal feedback control] *Suppose that all hypotheses of Theorem 1.1 hold. Let u_h^* be defined as (4.8) and w_h be the solution of (2.1). Then for any given $z \in \mathbb{R}^n$,*

$$w_h(z,0) \le J_{z,0}^h(u_h(\cdot)), \quad \forall\, u_h(\cdot) \in \Delta,$$

$$w_h(z,0) = J_{z,0}^h\left(u_h^*(\cdot)\right),$$

where $J_{z,0}^h$ is defined by (4.2).

Proof The first assertion follows from the following argument

$$\begin{aligned} w_h(z,0) &= \min_{u \in \mathbb{U}} \{ hL(z,0,u) + w_h(z + hf(z,0,u), s_1) \} \le hL(z,0,u_0) + w_h(y_1, s_1) \\ &= hL(y_0, s_0, u_0) + \min_{u \in \mathbb{U}} \{ hL(y_1, s_1, u) + w_h(y_1 + hf(y_1, s_1, u), s_2) \} \\ &\le hL(y_0, s_0, u_0) + hL(y_1, s_1, u_1) + w_h(y_2, s_2) \\ &\le \cdots \\ &\le \sum_{l=0}^{N-1} hL(y_l, s_l, u_l) + w_h(y_N, s_N) = J_{z,0}^h(u_h(\cdot)), \end{aligned} \tag{4.9}$$

where $u_l = u_h(s_l)$, $l = 0, 1, \ldots, N-1$, and y_l is the solution of (4.1) with $x = z$ and $s = j_{h,s} = 0$.

For the second assertion, we choose $u_h(\cdot) = u_h^*(\cdot)$. Then $u_l = u_l^* = m_h^*\left(y^l, s_l\right)$ and $y_l = y^l, l = 0, 1, \ldots, N-1$, where y^l is the solution of (4.7). By the definition of $m_h^*\left(y^l, s_l\right)$, all inequalities in (4.9) become equalities. The result then follows. □

Theorem 4.2 [Minimizing sequence of continuous optimal feedback control] *Suppose that all hypotheses of Theorem 1.1 hold. Let u_h^* be defined as (4.8). Then for any given $z \in \mathbb{R}^n$,*

$$\lim_{h\to 0^+} J_{z,0}\left(u_h^*(\cdot)\right) = \inf_{u(\cdot)\in\Delta} J_{z,0}\left(u(\cdot)\right) \equiv w(z, 0), \tag{4.10}$$

where $J_{z,0}$ is defined by (1.5).

Proof We claim that (4.10) is equivalent to saying that

$$\lim_{h\to 0^+} J_{z,0}\left(u_h^*(\cdot)\right) = \lim_{h\to 0^+} J_{z,0}^h\left(u_h^*(\cdot)\right) \equiv \lim_{h\to 0^+} w_h(z, 0) = w(z, 0). \tag{4.11}$$

Actually,

$$\begin{aligned}
&\left|J_{z,0}\left(u_h^*(\cdot)\right) - J_{z,0}^h\left(u_h^*(\cdot)\right)\right| \\
&\le \left|\int_0^T L\left(y^*(t), t, u_h^*(t)\right) \mathrm{d}t + \psi\left(y^*(T)\right) - \sum_{j=0}^{N-1} hL\left(y^j, s_j, u_j^*\right) - \psi\left(y^N\right)\right| \\
&\le \left|\int_0^T L\left(y^*(t), t, u_h^*(t)\right) \mathrm{d}t - \sum_{j=0}^{N-1} hL\left(y^j, s_j, u_j^*\right)\right| + \left|\psi(y^*(T)) - \psi\left(y^N\right)\right| \\
&\triangleq I_4 + I_5,
\end{aligned} \tag{4.12}$$

where $y^*(\cdot)$ is the solution of system (1.1) with control $u_h^*(\cdot)$ defined by (4.8), $u_j^* = m_h^*\left(y^j, s_j\right)$ is determined by (4.5)–(4.6) and y^j is the solution of system (4.7). The proof will be accomplished if we can show that

$$\left\|y^*(t) - y^j\right\| \le C_T h, \quad t \in [s_j, s_{j+1}],\ \forall\ j = 0, 1, \ldots, N-1. \tag{4.13}$$

In fact,

$$\begin{aligned}
I_4 &\le \sum_{j=0}^{N-1} \int_{s_j}^{s_{j+1}} \left|L\left(y^*(t), t, u_j^*\right) - L\left(y^j, s_j, u_j^*\right)\right| \mathrm{d}t \\
&\le \sum_{j=0}^{N-1} \int_{s_j}^{s_{j+1}} L_L\left(\left\|y^*(t) - y^j\right\| + |t - s_j|\right) \mathrm{d}t \le T L_L(C_T + 1)h,
\end{aligned} \tag{4.14}$$

and

$$I_5 = \left|\psi\left(y^*(T)\right) - \psi\left(y^N\right)\right| \le L_\psi \left\|y^*(T) - y^N\right\| \le L_\psi C_T h, \tag{4.15}$$

where C_T is a constant independent of h. (4.11) then follows from (4.12) and (4.14)–(4.15).

Now we check (4.13). Define

$$z^*(t) = y^j, \quad t \in [s_j, s_{j+1}),\ j = 0, 1, \ldots, N-1.$$

Since

$$\begin{cases} y^*(t) = y^*(s_j) + \int\limits_{s_j}^{t} f\left(y^*(\tau), \tau, u_j^*\right) \mathrm{d}\tau, \quad t \in [s_j, s_{j+1}], \\ y^{j+1} = y^j + hf\left(y^j, s_j, u_j^*\right), \quad j = 0, 1, \ldots, N-1, \end{cases}$$

and

$$\begin{aligned} \left\| y^*(s_j) - y^j \right\| &\le \left\| y^*(s_{j-1}) - y^{j-1} \right\| + \int\limits_{s_{j-1}}^{s_j} \left| f\left(y^*(\tau), \tau, u_{j-1}^*\right) - f\left(y^{j-1}, s_{j-1}, u_{j-1}^*\right) \right| \mathrm{d}\tau \\ &\le \left\| y^*(s_{j-1}) - y^{j-1} \right\| + L_f \int\limits_{s_{j-1}}^{s_j} \left\| y^*(\tau) - y^{j-1} \right\| \mathrm{d}\tau + L_f h^2, \ 1 \le j \le N, \end{aligned}$$

we have, for any $t \in [s_j, s_{j+1}),\ \ j \in \{0, 1, \ldots, N-1\}$, that

$$\begin{aligned} \|y^*(t) - z^*(t)\| &= \left\| y^*(t) - y^j \right\| \le \left\| y^*(t) - y^{j+1} \right\| + M_f h \\ &\le \left\| y^*(s_j) - y^j \right\| + \int\limits_{s_j}^{t} \left| f\left(y^*(\tau), \tau, u_j^*\right) - f\left(y^j, s_j, u_j^*\right) \right| \mathrm{d}\tau + 2M_f h \\ &\le \left\| y^*(s_j) - y^j \right\| + L_f \int\limits_{s_j}^{t} \left(\|y^*(\tau) - y_j\| + |\tau - s_j| \right) \mathrm{d}\tau + 2M_f h \\ &\le \left\| y^*(s_j) - y^j \right\| + L_f \int\limits_{s_j}^{t} \|y^*(\tau) - z^*(\tau)\| \, \mathrm{d}\tau + L_f h^2 + 2M_f h \\ &\le \left\| y^*(s_{j-1}) - y^{j-1} \right\| + L_f \int\limits_{s_{j-1}}^{t} \|y^*(\tau) - z^*(\tau)\| \, \mathrm{d}\tau + 2L_f h^2 + 2M_f h \\ &\le \ldots \\ &\le L_f \int\limits_{s_0}^{t} \|y^*(\tau) - z^*(\tau)\| \, \mathrm{d}\tau + (j+1) L_f h^2 + 2M_f h \\ &\le L_f \int\limits_{0}^{t} \|y^*(\tau) - z^*(\tau)\| \, \mathrm{d}\tau + \overline{C}_T h, \end{aligned} \tag{4.16}$$

where $\overline{C}_T = TL_f + 2M_f$. Apply the Gronwall's inequality to (4.16), to obtain (4.13):

$$\|y^*(t) - z^*(t)\| \le C_T h, \quad \forall\, t \in [s_j, s_{j+1}],\ \ j = 0, 1, \ldots, N-1$$

by setting $C_T = \overline{C}_T \mathrm{e}^{TL_f} = (TL_f + 2M_f)\mathrm{e}^{TL_f}$. The proof is complete. □

4.2 Time-space discretization case

Now let us consider the time-space discretization case. Recall that in Sect. 3, $\Omega \subset \mathbb{R}^n$ is assumed to be a closed bounded polyhedron such that for all sufficiently small h,

$$x + hf(x, s, u) \in \Omega, \quad \forall\, (x, s, u) \in \Omega \times [0, T] \times \mathbb{U}.$$

For $(x_i, s_j) \in G \times S$, $v_h^k(x_i, s_j)$ is defined by (3.2), while for $(x, s) \in \Omega \times [0, T]$, $w_h^k(x, s)$ is defined by (3.3), and $w_h^k = v_h^k$ on $G \times S$.

By (3.3), for $(x_i, s_j) \in G \times S$, we can rewrite (3.2) as

$$\begin{cases} w_h^k(x_i, s_j) = \min\limits_{u\in\mathbb{U}} \left\{hL(x_i, s_j, u) + w_h^k\left(x_i + hf(x_i, s_j, u), s_{j+1}\right)\right\}, \\ \qquad\qquad i = 1, 2 \ldots, M, \;\; j = 0, 1, \ldots, N-1, \\ w_h^k(x_i, T) = \psi(x_i), \quad i = 1, 2 \ldots, M. \end{cases} \tag{4.17}$$

Equations (2.1) and (4.17) motivate us to define P and P^k, respectively, as

$$P(x, s_j, u) \triangleq hL(x, s_j, u) + w_h\left(x + hf(x, s_j, u), s_{j+1}\right), \;\; (x, s_j, u) \in \Omega \times S \backslash \{T\} \times \mathbb{U}, \tag{4.18}$$

and

$$P^k(x, s_j, u) \triangleq hL(x, s_j, u) + w_h^k\left(x + hf(x, s_j, u), s_{j+1}\right), \;\; (x, s_j, u) \in \Omega \times S \backslash \{T\} \times \mathbb{U}. \tag{4.19}$$

Notice that w_h^k in the definition of P^k can be determined by (3.3) and (3.2).

The following Corollary 4.2 is a consequence of Theorem 3.1.

Corollary 4.2 *Suppose that all hypotheses of Theorem 1.1 hold. Let C be the constant in Theorem 3.1. Then*

$$\left|P^k - P\right| \le C\left(\frac{k}{h} + k\right) \to 0 \;\; \textit{uniformly on } \Omega \times S\backslash\{T\} \times \mathbb{U} \;\; \textit{as } k \to 0^+.$$

Next, for any $x \in \Omega$ and $j \in \{0, 1, \ldots, N-1\}$, it follows from the assumptions (1.8) that there exist at least one control $m_h^*(x, s_j) \in \mathbb{U}$ and one $m_k^*(x, s_j) \in \mathbb{U}$ such that

$$P\left(x, s_j, m_h^*(x, s_j)\right) = \min_{u\in\mathbb{U}} P(x, s_j, u) \equiv w_h(x, s_j) \quad \text{by (2.1)}, \tag{4.20}$$

and

$$P^k\left(x, s_j, m_k^*(x, s_j)\right) = \min_{u\in\mathbb{U}} P^k(x, s_j, u). \tag{4.21}$$

Consider the following subset of $\mathbb{U}$:

$$A_2(x, s_j) = \min_{\|\cdot\|} \left\{v \in \mathbb{U} \mid P^k\left(x, s_j, v\right) = \min_{u\in\mathbb{U}} P^k(x, s_j, u)\right\}. \tag{4.22}$$

This is a subset of controls with minimal energy that satisfy (4.21). The non-emptiness of $A_2(x, s_j)$ is guaranteed by assumptions (1.8). It is also remarked that $A_2(x, s_j)$ may not be a singleton, but we can choose any one element $a_2(x, s_j) \in A_2(x, s_j)$, to determine the minimal energy control function m_k^* by

$$m_k^*(x, s_j) = a_2(x, s_j), \quad x \in \Omega, \;\; j = 0, 1, \ldots, N-1. \tag{4.23}$$

In this way, the control m_k^* is well defined on $\Omega \times S\backslash\{T\}$.

Now, we can define a feedback control law u_k^* for time-space discretization system

$$u_k^*(s) = u_{k,j}^* \triangleq m_k^*\left(y^j, s_j\right), \quad s \in [s_j, s_{j+1}), \quad j = 0, 1, \ldots, N-1, \tag{4.24}$$

where again $y^j \approx y(s_j)$ denotes the solution of the following system with control u_k^* determined by (4.24) without diffusion:

$$\begin{cases} y^{j+1} = y^j + hf\left(y^j, s_j, u_k^*(s_j)\right), \quad j = 0, 1, \ldots, N-1, \\ y_0 = z, \end{cases} \tag{4.25}$$

where $z \in \Omega$ is given. Equation 4.25 is also the discrete counterpart for control system (1.1).

Theorem 4.3 [Minimizing sequence of time-discretization optimal feedback control] *Suppose that all hypotheses of Theorem 1.1 hold. Let u_k^* be defined as (4.24). Then for any given $z \in \Omega$,*

$$\left|J_{z,0}^h\left(u_k^*(\cdot)\right) - w_h(z,0)\right| \le 4TC\frac{k}{h^2} \to 0 \quad as\ k \to 0^+, \tag{4.26}$$

where $J_{z,0}^h$ is defined by (4.2) and C is the constant in Theorem 3.1.

Proof From (4.19), we have, for all $j = 0, 1, \ldots, N-1$ and $x \in \Omega$, that

$$\begin{aligned} hL\left(x, s_j, m_k^*(x, s_j)\right) = &-w_h^k\left(x + hf(x, s_j, m_k^*(x, s_j)), s_{j+1}\right) \\ &+ P^k\left(x, s_j, m_k^*(x, s_j)\right), \end{aligned} \tag{4.27}$$

where $m_k^*(x, s_j)$ is determined by (4.22)–(4.23). Set $x = y^j$ in (4.27), where y^j is the solution of system (4.25), and summarize j from 0 to N $-$ 1, to get

$$\sum_{j=0}^{N-1} hL\left(y^j, s_j, u_{k,j}^*\right) = \sum_{j=0}^{N-1}\left[-w_h^k\left(y^j + hf\left(y^j, s_j, u_{k,j}^*\right), s_{j+1}\right) + P^k\left(y^j, s_j, u_{k,j}^*\right)\right], \tag{4.28}$$

where $u_{k,j}^* = m_k^*\left(y^j, s_j\right)$.

By definitions of $u_j^* = m_h^*\left(y^j, s_j\right)$ in (4.8) that is characterized by (4.5)–(4.6) and $u_{k,j}^* = m_k^*\left(y^j, s_j\right)$ in (4.24) that is characterized by (4.22)–(4.23), respectively, we have

$$P\left(y^j, s_j, u_{k,j}^*\right) - P\left(y^j, s_j, u_j^*\right) \ge 0,$$
$$P^k\left(y^j, s_j, u_{k,j}^*\right) - P^k\left(y^j, s_j, u_j^*\right) \le 0.$$

In view of the facts above, on the one hand, it follows from (4.20), (4.28), Theorem 3.1 and Corollary 4.2, that

$$
\begin{aligned}
I_6 &\triangleq J_{z,0}^h\left(u_k^*(\cdot)\right) - w_h(z,0)\\
&= \sum_{j=0}^{N-1} hL\left(y^j, s_j, u_{k,j}^*\right) + \psi\left(y^N\right) - w_h(z,0)\\
&= \sum_{j=0}^{N-1}\left[-w_h^k\left(y^j + hf\left(y^j, s_j, u_{k,j}^*\right), s_{j+1}\right) + P^k\left(y^j, s_j, u_{k,j}^*\right)\right] + \psi\left(y^N\right)\\
&\quad - \left(\sum_{j=0}^{N-1}\left[w_h\left(y^j, s_j\right) - w_h\left(y^{j+1}, s_{j+1}\right)\right] + \psi\left(y^N\right)\right)\\
&= \sum_{j=0}^{N-1}\left\{\left[-w_h^k\left(y^{j+1}, s_{j+1}\right) + w_h\left(y^{j+1}, s_{j+1}\right)\right] + \left[P^k\left(y^j, s_j, u_{k,j}^*\right) - P\left(y^j, s_j, u_j^*\right)\right]\right\}\\
&= \sum_{j=0}^{N-1}\left\{\left[-w_h^k\left(y^{j+1}, s_{j+1}\right) + w_h\left(y^{j+1}, s_{j+1}\right)\right] + \left[P^k\left(y^j, s_j, u_{k,j}^*\right) - P\left(y^j, s_j, u_{k,j}^*\right)\right]\right.\\
&\quad \left. + \left[P\left(y^j, s_j, u_{k,j}^*\right) - P\left(y^j, s_j, u_j^*\right)\right]\right\}\\
&\geq \sum_{j=0}^{N-1}\left\{\left[-w_h^k\left(y^{j+1}, s_{j+1}\right) + w_h\left(y^{j+1}, s_{j+1}\right)\right] + \left[P^k\left(y^j, s_j, u_{k,j}^*\right) - P\left(y^j, s_j, u_{k,j}^*\right)\right]\right\}\\
&\geq -\sum_{j=0}^{N-1} 2C\left(\frac{k}{h} + k\right) = -2NC\left(\frac{k}{h} + k\right) = -2TC(1+h)\frac{k}{h^2} \geq -4TC\frac{k}{h^2},
\end{aligned}
\tag{4.29}
$$

and on the other hand,

$$
\begin{aligned}
I_6 &= \sum_{j=0}^{N-1}\left\{\left[-w_h^k\left(y^{j+1}, s_{j+1}\right) + w_h\left(y^{j+1}, s_{j+1}\right)\right] + \left[P^k\left(y^j, s_j, u_{k,j}^*\right) - P\left(y^j, s_j, u_j^*\right)\right]\right\}\\
&= \sum_{j=0}^{N-1}\left\{\left[-w_h^k\left(y^{j+1}, s_{j+1}\right) + w_h\left(y^{j+1}, s_{j+1}\right)\right] + \left[P^k\left(y^j, s_j, u_{k,j}^*\right) - P^k\left(y^j, s_j, u_j^*\right)\right]\right.\\
&\qquad \left. + \left[P^k\left(y^j, s_j, u_j^*\right) - P\left(y^j, s_j, u_j^*\right)\right]\right\}\\
&\leq \sum_{j=0}^{N-1}\left\{\left[-w_h^k\left(y^{j+1}, s_{j+1}\right) + w_h\left(y^{j+1}, s_{j+1}\right)\right] + \left[P^k\left(y^j, s_j, u_j^*\right) - P\left(y^j, s_j, u_j^*\right)\right]\right\}\\
&\leq \sum_{j=0}^{N-1} 2C\left(\frac{k}{h} + k\right) = 2NC\left(\frac{k}{h} + k\right) = 2TC(1+h)\frac{k}{h^2} \leq 4TC\frac{k}{h^2}.
\end{aligned}
\tag{4.30}
$$

Combine (4.29) and (4.30) to give the required result (4.26). The proof is complete. □

It is pointed out that in Theorem 1.7 of [15] on p. 479–480, the convergence was concluded by passing limit of k to zero directly in an equality similar to (4.28). This equality, using our functions, is of the form:

$$\sum_{j=0}^{N-1} hL\left(y^j, s_j, u^*_{k,j}\right) = \sum_{j=0}^{N-1} \left[-w_h^k\left(y^j + hf\left(y^j, s_j, u^*_{k,j}\right), s_{j+1}\right) + w_h^k\left(y^j, s_j\right)\right]. \tag{4.28$'$}$$

Unfortunately, (4.28′) can not be obtained by the arguments used in [15]. The reason is that it is admitted that

$$P^k\left(y^j, s_j, u^*_{k,j}\right) = w_h^k(y^j, s_j) \quad \text{on } \Omega \ni y^j,$$

and then replace $P^k\left(y^j, s_j, u^*_{k,j}\right)$ in (4.28) by $w_h^k(y^j, s_j)$ to get (4.28′). But in [15], $P^k\left(x_i, s_j, u^*_{k,j}\right) = w_h^k(x_i, s_j)$ is true only on node point set $G = \{x_i\}$ not on the whole Ω.

Notice that there is no time term s in [15] but the time term s has no influence. To sum up, since we can only obtain (4.28) instead of (4.28′), the "unique minimum assumption" of [15] is useless anymore. The key points to the result should be estimates (4.29) and (4.30), which can be used to correct the gap in [15] as well.

Theorem 4.4 [Minimizing sequence of continuous optimal feedback control] *Under the assumptions of Theorem 4.3, let $k = O(h^{2+\gamma})$ for some constant $\gamma > 0$, and w the viscosity solution of HJB equation (1.7). Then for any given $z \in \Omega$,*

$$J_{z,0}\left(u_k^*(\cdot)\right) \to \inf_{u(\cdot)\in\Delta} J_{z,0}\left(u(\cdot)\right) \equiv w(z, 0) \quad \text{as } h \to 0^+, \tag{4.31}$$

where $J_{z,0}$ is defined by (1.5).

Proof The proof is similar to that for Theorem 4.2. For the sake of completeness, here we give a detailed proof.

For any given $z \in \Omega$, we have

$$\begin{aligned}\left|J_{z,0}\left(u_k^*(\cdot)\right) - w(z,0)\right| &\le \left|J_{z,0}\left(u_k^*(\cdot)\right) - J_{z,0}^h\left(u_k^*(\cdot)\right)\right| \\ &\quad + \left|J_{z,0}^h\left(z, 0, u_k^*(\cdot)\right) - w_h(z,0)\right| + \left|w_h(z,0) - w(z,0)\right| \\ &\triangleq I_7 + I_8 + I_9.\end{aligned} \tag{4.32}$$

By (4.26), the term

$$I_8 \triangleq \left|J_{z,0}^h\left(u_k^*(\cdot)\right) - w_h(z,0)\right| \le 4TC\frac{k}{h^2} = O\left(h^\gamma\right) \to 0 \quad \text{as } h \to 0^+. \tag{4.33}$$

The fact that the term I_9 converges to 0 as $h \to 0^+$ is confirmed by Theorem 2.1. Hence, only the term I_7 needs to be addressed. Now,

$$\begin{aligned}I_7 &\triangleq \left|J_{z,0}\left(u_k^*(\cdot)\right) - J_{z,0}^h\left(u_k^*(\cdot)\right)\right| \\ &\le \left|\int_0^T L\left(y_k^*(t), t, u_k^*(t)\right) \mathrm{d}t + \psi\left(y_k^*(T)\right) - \sum_{j=0}^{N-1} hL\left(y^j, s_j, u^*_{k,j}\right) - \psi\left(y^N\right)\right|\end{aligned}$$

$$\leq \left| \int_0^T L\left(y_k^*(t), t, u_k^*(t)\right) \mathrm{d}t - \sum_{j=0}^{N-1} hL\left(y^j, s_j, u_{k,j}^*\right) \right| + \left| \psi\left(y_k^*(T)\right) - \psi\left(y^N\right) \right|$$

$$\triangleq I_{71} + I_{72}, \tag{4.34}$$

where $y_k^*(\cdot)$ is the solution of system (1.1) with control $u_k^*(\cdot)$ defined by (4.24), $u_{k,j}^* = m_k^*\left(y^j, s_j\right)$ is specified by (4.22)–(4.23), and again y^j is the solution of system (4.25). Similar to the estimate (4.13), we also have

$$\left\| y_k^*(t) - y^j \right\| \leq C_T h, \quad t \in [s_j, s_{j+1}], \ \forall\, j = 0, 1, \ldots, N-1.$$

Therefore,

$$I_{71} \leq \sum_{j=0}^{N-1} \int_{s_j}^{s_{j+1}} \left| L\left(y_k^*(t), t, u_{k,j}^*\right) - L\left(y^j, s_j, u_{k,j}^*\right) \right| \mathrm{d}t$$

$$\leq \sum_{j=0}^{N-1} \int_{s_j}^{s_{j+1}} L_L\left(\left\| y_k^*(t) - y^j \right\| + |t - s_j| \right) \mathrm{d}t \leq T L_L (C_T + 1) h, \tag{4.35}$$

and

$$I_{72} = \left| \psi\left(y_k^*(T)\right) - \psi\left(y^N\right) \right| \leq L_\psi \left\| y_k^*(T) - y^N \right\| \leq L_\psi C_T h. \tag{4.36}$$

The assertion (4.31) then follows from (4.32)–(4.36). □

5 Approximation algorithm

We now present an algorithm which summarizes the procedure for optimal feedback control approximation developed in the preceding sections.

Note that only the full discretization scheme (3.2) can be used in numerical computation, whereas the scheme (2.1) is of theoretical significance only since the space variable therein is not in discrete form. The algorithm calculates the viscosity solution and optimal control-trajectory pairs.

5.1 The algorithm

Based on the full discretization scheme (3.2) and convex linear combination scheme (3.3), we can now present the algorithm as follows.

Step 1: Initial partitioning on time and space. Let $N > 0$ be a fixed integer. Let $S = \{s_j = jh \mid j = 0, 1, \ldots, N\}$ define an equal-partition of $[0, T]$ with interval length $0 < h = T/N < 1$. For $q \in \{0, 1, \ldots, N\}$, denote by (u^q, y^q) the q-th optimal control-trajectory pair.
The domain $\Omega \subset \mathbb{R}^n$ is assumed to be a closed bounded polyhedron such that for all sufficiently small $h > 0$ and all $(x, s, u) \in \Omega \times [0, T] \times \mathbb{U}$, it holds that $x + hf(x, s, u) \in \Omega$. Let $k = Ch^{2+\gamma}$ be the size of space-discretization mesh, where $C, \gamma > 0$ are arbitrarily appropriately given constants. Construct a regular triangulation of Ω that consists of a finite number of simplices $\{T_i\}$ such that

$\Omega = \cup_i T_i$ and $\max_i\{\text{diam}(T_i)\} = k$. For numerical purpose, the regular triangulation process could be completed once the size k is given.
Denote by x_l, $l = 1, 2, \ldots, M$, the nodes of the triangulation. Set $G = \{x_l |\ l = 1, 2, \ldots, M\}$.

Step 2: Calculating approximate viscosity solution. This is done by the finite difference scheme in time-space mesh approximation via (3.2):

$$\begin{cases} V_i^j = hL_i^j + \sum_{l=1}^{M} \lambda_i^{l,j} V_l^{j+1}, \\ u_i^j \in \arg\inf_{u\in\mathbb{U}} \left\{Y_i^j(u)\right\}, \quad \text{where } Y_i^j(u) \triangleq hL(x_i, s_j, u) + \sum_{l=1}^{M} \lambda_i^l(u, s_j) V_l^{j+1}, \\ V_i^N = \psi(x_i) \end{cases} \tag{5.1}$$

for $i = 1, 2, \ldots, M$ and $j = N-1, N-2, \ldots, 1, 0$, where

$$V_i^j = V(x_i, s_j), \quad L_i^j = L\left(x_i, s_j, u_i^j\right), \quad \lambda_i^{l,j} = \lambda_i^l\left(u_i^j, s_j\right)$$

and $\left\{\lambda_i^l(u, s_j) \in [0, 1],\ l = 1, 2, \ldots, M\right\}$ is the set of coefficients in the unique convex linear combination $x_i + hf(x_i, s_j, u) = \sum_{l=1}^{M} \lambda_i^l(u, s_j) x_l$ with $\sum_{l=1}^{M} \lambda_i^l (u, s_j) = 1$. This convex linear combination is produced in terms of the vertices of the simplex where $x_i + hf(x_i, s_j, u)$ is located.

More precisely, since $\left\{V_i^N\right\}_{i=1}^M$ is given, when $\left\{V_i^{j+1}\right\}_{i=1}^M$ is known, then for any $i \in \{1, 2, \ldots, M\}$, V_i^j can be solved via (5.1) by the following sub-steps:

Step 2.1: Discretize the control domain $\mathbb{U}$ as a finite set $\mathbb{U}_d \triangleq \{u_l :\ l=0, 1, 2, \ldots\}$ at a given size of control-space mesh $d > 0$.

Step 2.2: For a fixed $u \in \mathbb{U}_d$, using a searching technique in numerical analysis, one can find the simplex T_{i_0} where $x_i + hf(x_i, s_j, u)$ is located. By producing the convex linear combination for $x_i + hf(x_i, s_j, u)$ in terms of the vertices of simplex T_{i_0}, one can obtain the coefficients $\{\lambda_i^l(u, s_j)\}$.

Step 2.3: Calculate $Y_i^j(u)$ for all $u \in \mathbb{U}_d$, and set

$$V_i^j = \min_{v\in\mathbb{U}_d} \left\{Y_i^j(v)\right\}, \quad A \triangleq \min_{\|\cdot\|} \left\{u \in \mathbb{U}_d \ |\ Y_i^j(u) = V_i^j\right\}.$$

It is obvious that $V_i^j = Y_i^j\left(u_i^j\right)$ for any $u_i^j \in A$.

Repeat Steps 2.1–2.3 above for all $i \in \{1, 2, \ldots, M\}$ from $j = N-1$ to $j = 0$ to complete Step 2 and obtain the approximate viscosity solution on $G \times S$: $\left\{\left\{V_i^j\right\}_{i=1}^M\right\}_{j=0}^N$.

It is worth noting that mathematical programming is not used to solve the optimization problem (3.2).

Step 3: Calculating the optimal feedback control-trajectory pairs $\{(u^q, y^q)\}_{q=0}^N$. For initial setting, let $y^0 = z$ and set $q = 0$.

Step 3.1: Calculating the q-th optimal feedback control u^q.

Since y^q and $\left\{\left\{V_i^j\right\}_{i=1}^M\right\}_{j=0}^N$ are known, the q-th optimal feedback control $u^q = u(s_q)$ is determined by

$$\begin{cases} u^q \in \arg\inf_{u\in\mathbb{U}} \left\{hL\left(y^q, s_q, u\right) \right. \\ \qquad \left. + W\left(y^q + hf\left(y^q, s_q, u\right), s_{q+1}\right)\right\}, \\ W\left(y^q + hf\left(y^q, s_q, u\right), s_{q+1}\right) \\ \quad = \sum_{l=1}^{M} \mu_q^l(u, s_q) V_l^{q+1} \quad \text{via (3.3)}, \end{cases} \tag{5.2}$$

where $\left\{\mu_q^l(u, s_q) \in [0, 1],\ l = 1, 2, \ldots, M\right\}$ is the set of coefficients in the unique convex linear combination $y^q + hf\left(y^q, s_q, u\right) = \sum_{l=1}^M \mu_q^l(u, s_q)x_l$ with $\sum_{l=1}^M \mu_q^l(u, s_q) = 1$. Here again the convex linear combination is produced in terms of the vertices of the simplex where $y^q + hf\left(y^q, s_q, u\right)$ is located.

Note that in (5.2), the control u^q is also chosen as the one with minimal energy. It is remarked that such a control with minimal energy may not be unique, but we can choose any one of them.

The calculation details are similar to those in Step 2.

After these steps, the q-th optimal feedback control-trajectory pair $(u^q, y^q) = \left(u(s_q), y(s_q)\right)$ is obtained.

Step 3.2: Calculating the $(q+1)$-th optimal trajectory y^{q+1}.

Once $u^q = u(s_q)$ is known, solve the state equation:

$$\frac{y^{q+1} - y^q}{h} = f(y^q, s_q, u^q)$$

to obtain the $(q+1)$-th optimal trajectory $y^{q+1} = y(s_{q+1})$.

Step 3.3: Iterating for the next time instant. Let $q = q + 1$. If $q = N$, then $u^q = u^{q-1}$, and end the procedure. Otherwise, go to Step 3.1. Repeat the iteration to get all optimal feedback control-trajectory pairs:

$$\left\{\left(u^q, y^q\right)\right\}_{q=0}^N = \left\{\left(u(s_q), y(s_q)\right)\right\}_{q=0}^N.$$

5.2 An example

We use the algorithm in Subsection 5.1 to solve the following optimal control problem:

$$\begin{cases} y'(t) = 2(1 - y(t))u(t), \quad t \in (0, 1], \\ y(0) = z, \\ J(u(\cdot)) = \int_0^1 |1 - y(t)|(1+t)^2 u(t)^2 \, dt + 2|1 - y(1)|, \\ \min_{u(\cdot)\in\Delta^*} J(u(\cdot)), \quad \text{where } \Delta^* \triangleq L^\infty([0, 1]; \mathbb{U}) \quad \text{and} \quad \mathbb{U} \triangleq [0, 1], \end{cases} \tag{5.3}$$

where $z \in \mathbb{R}$ is a given initial value.

This system is considered since its optimal control is unique and can be obtained analytically. Thus, we can calculate the numerical results via the algorithm.

The HJB equation corresponding to (5.3) is given as

$$\begin{cases} -w_s(x,s) - \inf_{u\in\mathbb{U}} \left\{ \nabla_x w(x,s)\cdot 2(1-x)u + |1-x|(1+s)^2u^2 \right\} = 0, \\ \qquad (x,s)\in\mathbb{R}\times[0,1), \\ w(x,1) = 2|1-x|,\ x\in\mathbb{R}. \end{cases} \tag{5.4}$$

It admits a unique viscosity solution according to the uniqueness theorem [36].

It is easy to check that the function $w \in C(\mathbb{R}\times[0,1])$ defined by

$$w(x,s) = |1-x|(1+s), \quad (x,s)\in\mathbb{R}\times[0,1] \tag{5.5}$$

is a viscosity solution of (5.4). Notice that when $z = 1$, the control problem (5.3) becomes a trivial case that $y(u(\cdot),\cdot)\equiv 1$ on $[0,1]$ and $J(u(\cdot)) = 0$ for any control function $u(\cdot)\in\Delta^*$. We consider only the case of $z\neq 1$. In this case, the unique optimal feedback control $u^*(\cdot)$ and the corresponding optimal trajectory $y^*(\cdot)$ of the system (5.3) are given analytically by

$$u^*(z,t) = u^*\left(y^*(z,t),t\right) = \frac{1}{1+t}, \quad t\in[0,1], \tag{5.6}$$

and

$$y^*(z,t) = \frac{t^2+2t+z}{(1+t)^2}, \quad t\in[0,1]. \tag{5.7}$$

To apply the algorithm, we should choose suitable space domain Ω that satisfies assumption (3.1), i.e., for all sufficiently small h, $x + hf(x,s,u)\in\Omega,\ \forall\ (x,s,u)\in\Omega\times[0,1]\times[0,1]$. For our example, where $\Omega = [0,1]$, assumption (3.1) is valid if $h < 1/2$. Since for any $(x,s,u)\in[0,1]\times[0,1]\times[0,1]$, we have

$$1-2hu > 0, \quad x + hf(x,s,u) = x + 2h(1-x)u = (1-2hu)x + 2hu \geq 0,$$

and

$$x + hf(x,s,u) = x + 2h(1-x)u = (1-2hu)x + 2hu \leq 1 - 2hu + 2hu = 1.$$

Let $h = 0.01$, and $k = d = 0.005$. Using the algorithm, we obtain numerically the viscosity solution and the optimal feedback control-trajectory pairs of control problem (5.3). The results are presented in Figs. 1, 2, and 3 for two different initial values ($z = 0.30$ and $z = 0.65$). The algorithm was implemented using MATLAB programming language.

Figure 1 displays the numerical solution of the viscosity solution (5.5). Figures 2 and 3 show the numerical solutions of the optimal feedback control and trajectory functions (5.6) and (5.7) with the initial state $z = 0.3$ and $z = 0.65$, respectively. In Table 1, we list the computed errors in the maximum norm between the numerical solutions and analytical solutions of w, u^* and y^*. We see that the numerical solutions obtained by the algorithm are very close to the analytical solutions. The comparison indicates the effectiveness of the algorithm we proposed.

It should be pointed out that not all the conditions of Theorem 1.1 are satisfied in this example. More precisely, the boundedness property for f, L, ψ in (1.8) are not satisfied. Nevertheless, the numerical result is found to be still satisfactory. The conditions are placed for the sake of mathematical rigor in the proof of Theorem 1.1. The simulation results suggest that our algorithm may be applicable to a larger class of systems.

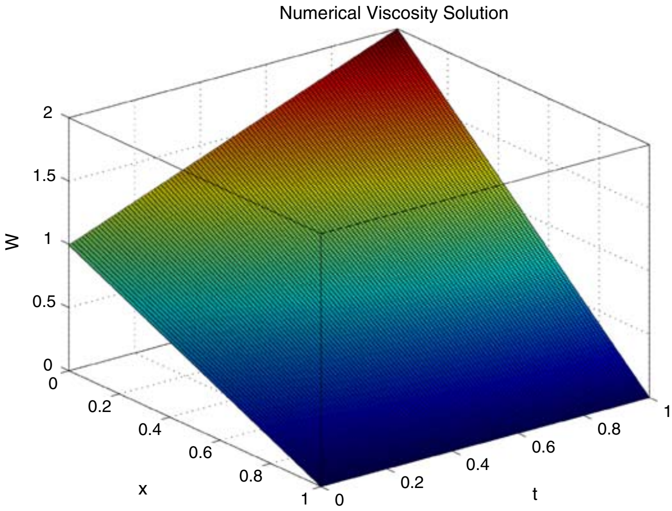

Fig. 1 Numerical solution of the viscosity solution

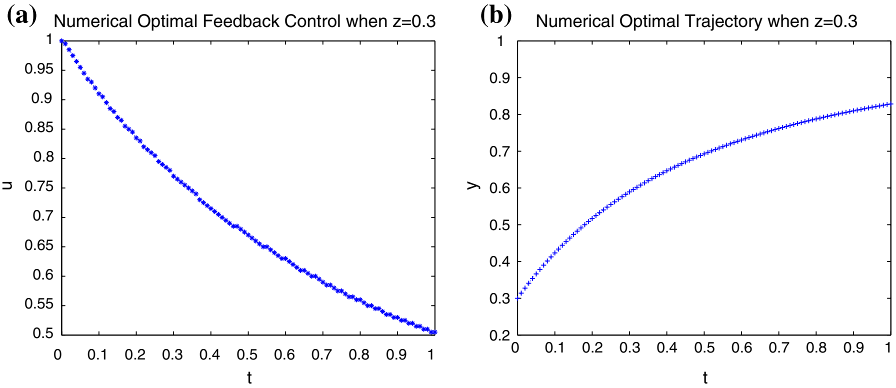

Fig. 2 Numerical solutions of optimal feedback control and trajectory functions with $z = 0.3$

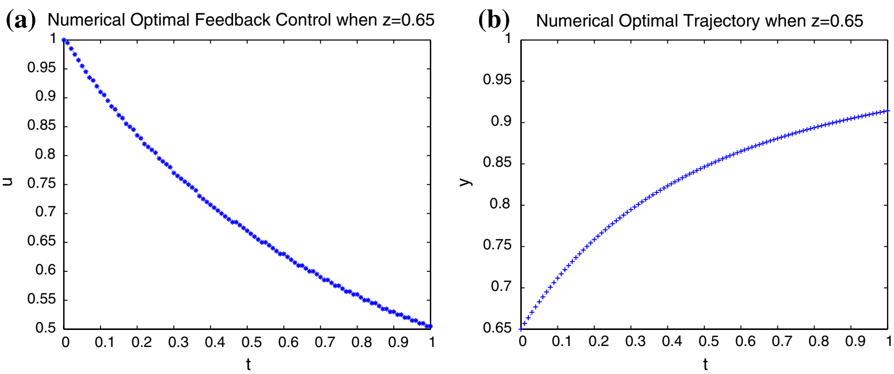

Fig. 3 Numerical solutions of optimal feedback control and trajectory functions with $z = 0.65$

Table 1 The computed errors in the maximum norm between the numerical solutions and analytical solutions for w, u^* and y^*

Error	Viscosity solution	Control	Trajectory
$z = 0.3$	7.5e−3	3.9e−3	5.0e−3
$z = 0.65$	7.5e−3	1.9e−3	5.0e−3

6 Concluding remarks and future works

We design two discretization schemes that calculate approximations to the viscosity solution of the evolutive HJB equation satisfied by the value function of a general continuous finite-dimensional control system with finite horizon cost functional. One scheme is based on the time difference approximation and the other on the time-space approximation. The convergence result is established for each scheme by the corresponding discrete dynamic programming. We show that the optimal control obtained from each scheme is "almost optimal" to the continuous system.

The merit of this work lies in its success in establishing convergence of the algorithm that leads to approximation of the optimal feedback control by dynamic programming for the problem with finite horizon cost functional without discount factor, while in literature, the results of the approximation of optimal feedback control are available only for infinite horizon cost functional problems. Moreover, for the approximation of the HJB equation, the discount factor in the cost functional that is key for convergence in literature is removed. An immediate issue then is to design such a simple algorithm that can be easily applied in practical optimal control problems. It is also worth investigating the relationship between the algorithm designed in this paper and the existing simple algorithms (such as the one in [21]) for which there is no convergence result available yet.

Acknowledgments The authors would like to thank anonymous referees for their careful reading and constructive suggestions that improve substantially the revision of the manuscript. The authors are very grateful to Professor Runyi Yu of Eastern Mediterranean University for many helps in revising the manuscript.

References

1. Bardi, M.: Some applications of viscosity solutions to optimal control and differential games. In: Dolcetta, I.C., Lions, P.L. (eds.) Viscosity Solutions and Applications, Lecture Notes in Mathematics, vol. 1660, pp. 44–97. Springer, Berlin (1997)
2. Bardi, M., Dolcetta, I.C.: Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations. Birkhauser, Boston (1997)
3. Barles, G., Souganidis, P.E.: Convergence of approximation schemes for fully nonlinear second order equations. J. Asymptot. Anal. **4**, 271–283 (1991)
4. Barron, E.N.: Application of viscosity solutions of infinite-dimensional Hamilton–Jacobi–Bellman equations to some problems in distributed optimal control. J. Optim. Theory Appl. **64**, 245–268 (1990)
5. Bryson, A.E. Jr.: Optimal control—1950–1985. IEEE Control Syst. Mag. **13**, 26–33 (1996)
6. Cannarsa, P., Gozzi, F., Soner, H.M.: A dynamic programming approach to nonlinear boundary control problems of parabolic type. J. Funct. Anal. **117**, 25–61 (1993)
7. Capuzzo Dolcetta, I.: On a descrete approximation of the Hamilton–Jacobi–Bellman equation of dynamic programming. Appl. Math. Optim. **10**, 366–377 (1983)
8. Capuzzo Dolcetta, I., Ishii, H.: Approximation solutions of the Bellman equation of deterministic control theory. Appl. Math. Optim. **11**, 161–181 (1984)
9. Crandall, M.G.: Viscosity solutions: a primer. In: Dolcetta, I.C., Lions, P.L. (eds.) Viscosity Solutions and Applications, Lecture Notes in Mathematics, vol. 1660, pp. 1–43. Springer, Berlin (1997)
10. Crandall, M.G., Evans, L.C., Lions, P.L.: Some properties of viscosity solutions of Hamilton–Jacobi equations. Tran. Amer. Math. Soc. **282**, 487–502 (1984)

11. Crandall, M.G., Ishii, H., Lions, P.L.: User's guide to viscosity solutions of second order partial differential equations. Bull. Amer. Math. Soc. **27**, 1–67 (1992)
12. Crandall, M.G., Lions, P.L.: Viscosity solutions of Hamilton–Jacobi equations. Tran. Amer. Math. Soc. **277**, 1–42 (1983)
13. Crandall, M.G., Lions, P.L.: Two approximations of solutions of Hamilton–Jacobi equations. Math. Comp. **43**, 1–19 (1984)
14. Falcone, M.: A numerical approach to the infinite horizon problem of deterministic control theory. Appl. Math. Optim. **15**, 1–13 (1987) and **23**, 213–214 (1991)
15. Falcone, M.: Numerical solutions of dynamic programming equations, Appendix in the book by Bardi, M., Capuzzo Dolcetta, I. (eds.) Optimal Control and Viscosity Solutions of Hamilton–Jacobi–Bellman Equations. Birkhauser, Boston (1997)
16. Falcone, M., Giorgi, T.: An approximation scheme for evolutive Hamilton–Jacobi equations. In: Stochastic Analysis, Control, Optimization and applications, Systems and Control: Foundations and Applications, pp. 289–303. Birkhauser, Boston (1999)
17. Fleming, W.H., Sonor, H.M.: Controlled Markov Processes and Viscosity Solutions. 2nd edn. Springer, New York (2006)
18. Gozzi, F., Sritharan, S.S., Swiech, A.: Viscosity solutions of dynamic-programming equations for the optimal control of the two-dimensional Navier–Stokes equations. Arch. Ration. Mech. Anal. **163**, 295–327 (2002)
19. Guo, B.Z., Sun, B.: Numerical solution to the optimal birth feedback control of a population dynamics: viscosity solution approach. Optim. Control Appl. Meth. **26**, 229–254 (2005)
20. Guo, B.Z., Sun, B.: Numerical solution to the optimal feedback control of continuous casting process. J. Glob. Optim. **39**, 171–195 (2007)
21. Guo, B.Z., Sun, B.: A new algorithm for finding numerical solutions of optimal feedback control. IMA Math.Control Inf. (to appear)
22. Huang, C.S., Wang, S., Teo, K.L.: Solving Hamilton–Jacobi–Bellman equations by a modified method of characteristics. Nonlinear Anal., TMA **40**, 279–293 (2000)
23. Huang, C.S., Wang, S., Teo, K.L.: On application of an alternating direction method to Hamilton–Jacobi–Bellman equations. J. Comput. Appl. Math. **166**, 153–166 (2004)
24. Kocan, M., Soravia, P.: A viscosity approach to infinite-dimensional Hamilton–Jacobi equations arising in optimal control with state constraints. SIAM J. Control Optim. **36**, 1348–1375 (1998)
25. Lions, P.L.: Generalized Solutions of Hamilton–Jacobi Equations. Pitman, London (1982)
26. Loxton, R.C., Teo, K.L., Rehbock, V.: Optimal control problems with multiple characteristic time points in the objective and constraints. Automatica **44**, 2923–2929 (2008)
27. Rubio, J.E.: Control and Optimization: The Linear Treatment of Nonlinear Problems, Nonlinear Science: Theory and Applications. Manchester University Press, Manchester (1986)
28. Souganidis, P.E.: Approximation schemes for viscosity solutions of Hamilton–Jacobi equations. J. Differ. Equ. **59**, 1–43 (1985)
29. Stoer, J., Bulirsch, R.: Introduction to numerical analysis. 2nd edn. Springer, New York (1993)
30. Stryk von, O., Bulirsch, R.: Direct and indirect methods for trajectory optimization. Annu. Oper. Res. **37**, 357–373 (1992)
31. Sussmann, H.J., Willems, J.C.: 300 years of optimal control: from the brachystochrone to the maximum principle. IEEE Controls Syst. Mag. **17**, 32–44 (1997)
32. Teo, K.L., Goh, C.J., Wong, K.H.: A Unified Computational Approach to Optimal Control Problems. Longman Scientific and Technical, England (1991)
33. Wang, S., Gao, F., Teo, K.L.: An upwind finite-difference method for the approximation of viscosity solutions to Hamilton–Jacobi–Bellman equations. IMA J. Math. Control. Inf. **17**, 167–178 (2000)
34. Wang, S., Jennings, L.S., Teo, K.L.: Numerical solution of Hamilton–Jacobi–Bellman equations by an upwind finite volume method. J. Glob. Optim. **27**, 177–192 (2003)
35. Willamson, W.E.: Use of polynomial approximations to calculate suboptimal controls. AIAA J. **9**, 2271–2273 (1971)
36. Yong, J.M.: Dynamic Programming Principle and Hamilton–Jacobi–Bellman Equations (in Chinese). Shanghai Scientific and Technical Publishers, Shanghai (1992)
37. Yong, J.M., Zhou, X.Y.: Stochastic Controls: Hamiltonian Systems and HJB Equations. Applications of Mathematics, vol. 43. Springer, New York (1999)